

## Fourier Transform

- \* Spectrum (or Frequency representation) of periodic signal is obtained by Fourier Series (FS), where both Magnitude spectrum and phase spectrum are Discrete function of frequency - They are line spectra.

The frequency spacing (or frequency interval) between the successive lines is equal to  $\frac{1}{T} = F_0$  where  $T$  is period of the signal, and  $F_0$  is called fundamental frequency or harmonic.

- \* For aperiodic (i.e. Energy and some power) signals, signal spectrum is obtained by Fourier Transform (FT), as follows.

$$\left\{ \begin{array}{l} x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \quad -\infty < t < \infty \\ \text{and} \\ X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad -\infty < \omega < \infty \end{array} \right.$$

where

$\omega = 2\pi F$  is radian/sec.

and  $F$  is frequency in Hz.

Since  $\omega = 2\pi F$ ,  $d\omega = 2\pi dF$ , we can also write:

$$\begin{cases} x(t) = \int_{F=-\infty}^{\infty} X(F) e^{j2\pi F t} dF, & -\infty < t < \infty \\ X(F) = \int_{t=-\infty}^{\infty} x(t) e^{-j2\pi F t} dt, & -\infty < F < \infty \end{cases}$$

Notations

$$x(t) \xleftrightarrow{FT} X(\omega) \text{ or } X(F)$$

$$X(F) \text{ or } X(\omega) = \mathcal{F}\{x(t)\}$$

Note

$$X(\omega) = |X(\omega)| e^{j\phi(\omega)}$$

$|X(\omega)|$  (or  $X(F)$ ) is Magnitude Spectrum

$\phi(\omega)$  (or  $\phi(F)$ ) is phase spectrum

Both  $|X(\omega)|$  and  $\phi(\omega)$  are Continuous functions of frequency,  $\omega$ .

\* If  $x(t)$  is real then

$$|X(\omega)| = |X(-\omega)| \quad \text{i.e. even function of frequency.}$$

and

$$\phi(\omega) = -\phi(-\omega) \quad \text{i.e. odd function of frequency.}$$

\* If  $x(t)$  is real and even in time, then

$X(\omega)$  is real function of frequency.

and still

$$|X(\omega)| = |X(-\omega)|, \quad \phi(\omega) = -\phi(-\omega)$$

Now

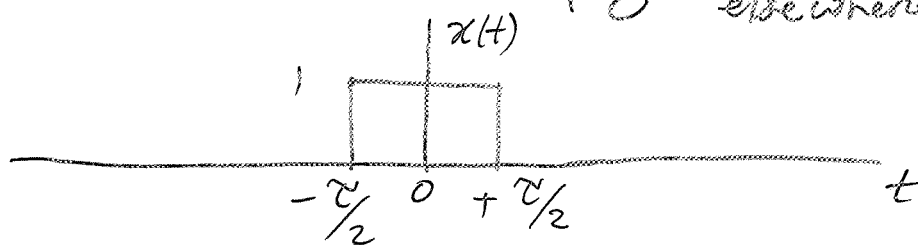
$$\phi(\omega) = \begin{cases} 0 & \text{if } X(\omega) > 0 \text{ for any } \omega \\ \pm\pi & \text{if } X(\omega) < 0 \text{ for any } \omega \\ \text{undefined} & \text{if } X(\omega) = 0 \text{ for any } \omega. \end{cases}$$

\* Condition for Existence of FT( $X(\omega)$ ) for  $x(t)$ :

(1)  $\int_{-\infty}^{\infty} |x(t)| dt < \infty$ , i.e. Absolutely Integrable.

(2) Over any time interval  $x(t)$  has finite number of maxima and minima

Ex 1 Given  $x(t) = \begin{cases} 1 & \text{for } -\tau/2 < t < \tau/2 \\ 0 & \text{elsewhere} \end{cases}$



Find  $X(\omega)$ .

Solution

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

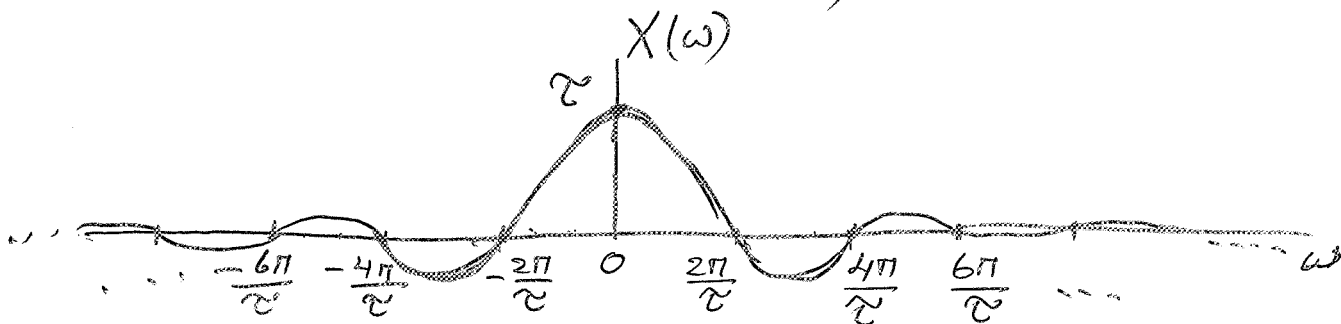
$$X(\omega) = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt = \frac{-1}{j\omega} \left\{ e^{-j\omega t} \right\}_{t=-\tau/2}^{\tau/2}$$

$$X(\omega) = \frac{-1}{j\omega} \left\{ e^{-j\omega \frac{\tau}{2}} - e^{+j\omega \frac{\tau}{2}} \right\}$$

$$X(\omega) = \frac{-1}{j\omega} \left\{ -2j \sin\left(\frac{\omega\tau}{2}\right) \right\}$$

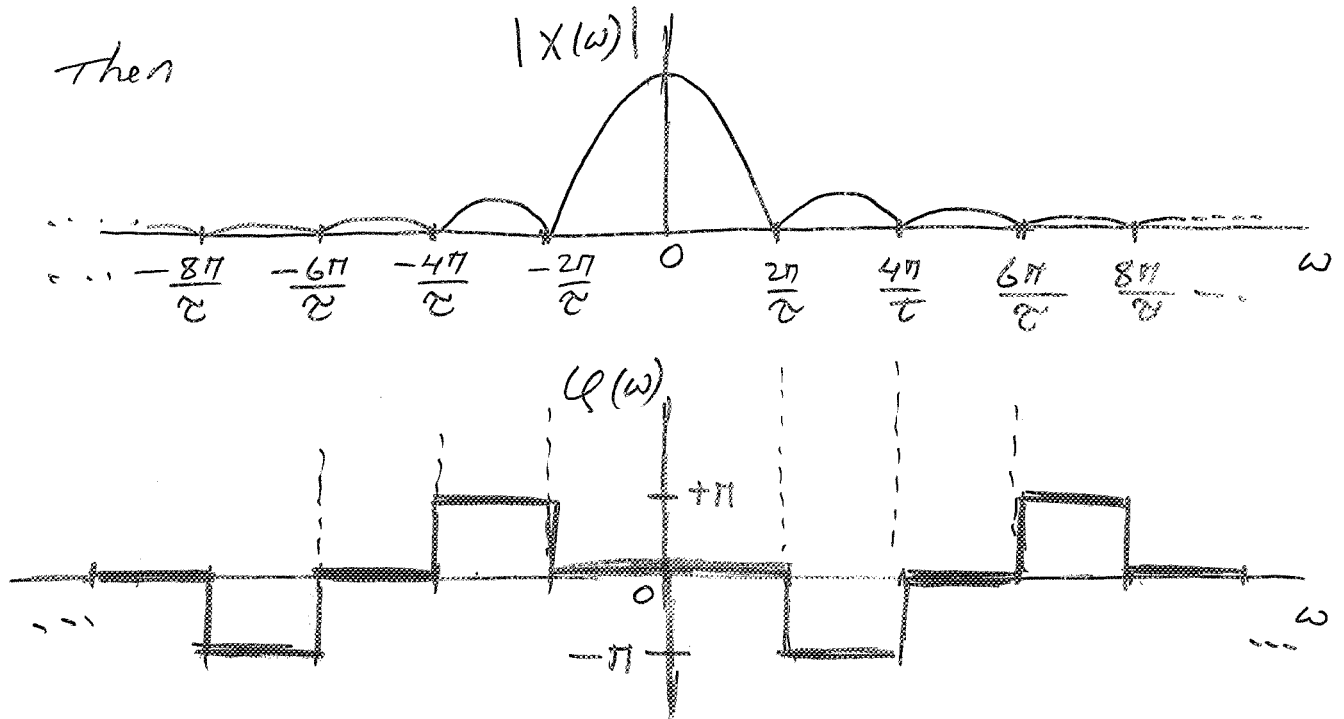
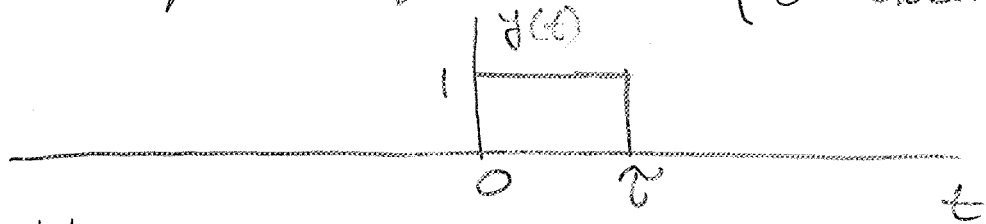
$$X(\omega) = \frac{2}{\omega} \sin\left(\frac{\omega\tau}{2}\right) = \frac{\tau}{\frac{\omega\tau}{2}} \sin\left(\frac{\omega\tau}{2}\right)$$

$$X(\omega) = \tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$$



$$\sin(k\pi) = 0 \Rightarrow \frac{\omega\tau}{2} = k\pi \text{ for } k = \pm 1, \pm 2, \dots$$

Then

Ex 2Find spectrum of  $y(t) = \begin{cases} 1 & 0 < t < \tau \\ 0 & \text{elsewhere} \end{cases}$ we obtain  $Y(\omega)$  as

$$Y(\omega) = \int_{t=0}^{\tau} e^{-j\omega t} dt = \frac{-1}{j\omega} \left\{ e^{-j\omega\tau} - 1 \right\}$$

$$Y(\omega) = \frac{-1}{j\omega} e^{-j\frac{\omega\tau}{2}} \left\{ e^{-j\frac{\omega\tau}{2}} - e^{+j\frac{\omega\tau}{2}} \right\}$$

$$Y(\omega) = e^{-j\frac{\omega\tau}{2}} \left( \frac{\tau}{\omega} \right) \sin\left(\frac{\omega\tau}{2}\right)$$

$$Y(\omega) = e^{-j\frac{\omega\tau}{2}} \tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right) = |Y(\omega)| e^{j\Theta(\omega)}$$

where comparing with those in Ex 1, we see that

$$(1) \quad |Y(\omega)| = \tau \left| \text{sinc}\left(\frac{\omega\tau}{2}\right) \right| = |X(\omega)|$$

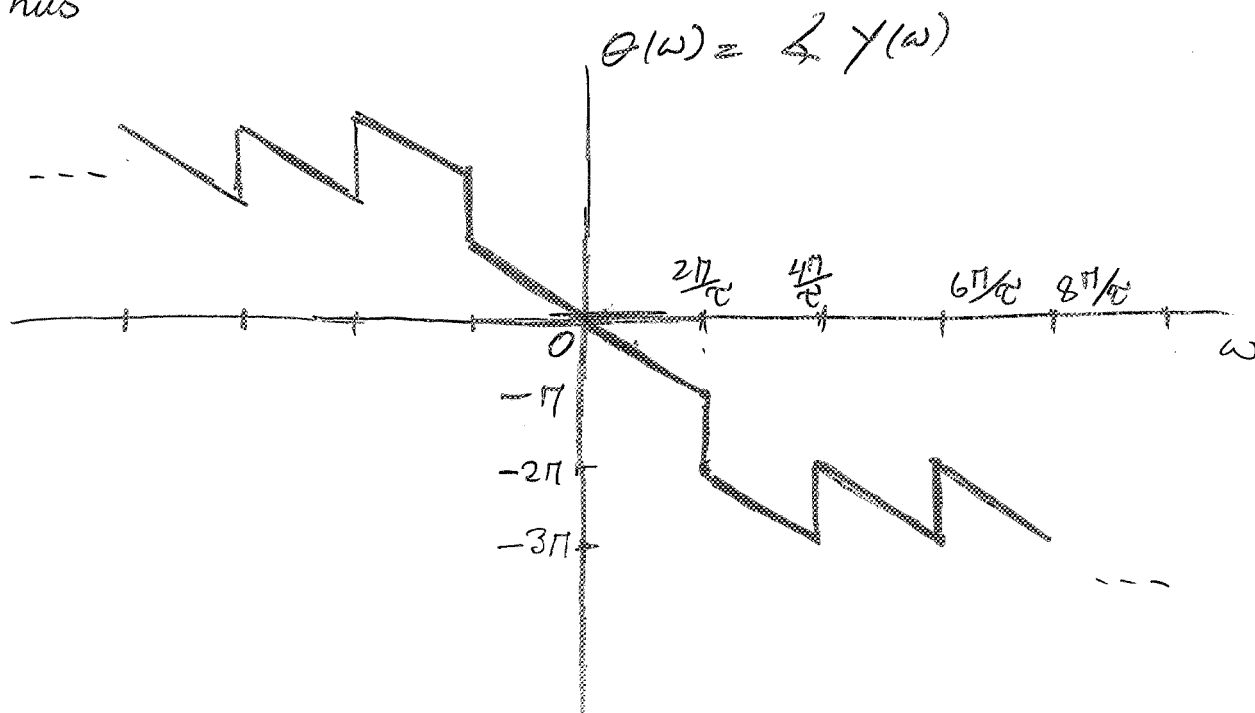
$$(2) \quad \theta(\omega) = \phi(\omega) - \frac{\omega\tau}{2}$$

That is  $x(t - \frac{\tau}{2}) = y(t)$

Thus  $Y(\omega) = e^{-j\omega\frac{\tau}{2}} X(\omega)$ ,

i.e. A linear time shift of  $\frac{\tau}{2}$  resulted in a linear frequency shift of  $-\frac{\omega\tau}{2}$  in the spectrum.

Thus



Ex 3

$$x(t) = e^{-\alpha t} u(t), \quad \alpha > 0$$

Find spectrum  $X(\omega)$  of  $x(t)$ .

\* First to see if  $\int_{-\infty}^{\infty} |x(t)| dt < \infty$ .

$$\begin{aligned} \int_{-\infty}^{\infty} |e^{-\alpha t}| dt &= \int_0^{\infty} |e^{-\alpha}|^t dt = \frac{1 - |e^{-\alpha}|^{\infty}}{1 - |e^{-\alpha}|} \\ &= \frac{1}{1 - |e^{-\alpha}|} < \infty \checkmark \end{aligned}$$

$$\begin{aligned} * X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{-\alpha t} u(t) e^{-j\omega t} dt \\ X(\omega) &= \int_0^{\infty} e^{-\alpha t} e^{-j\omega t} dt = \int_0^{\infty} e^{-(j\omega + \alpha)t} dt \end{aligned}$$

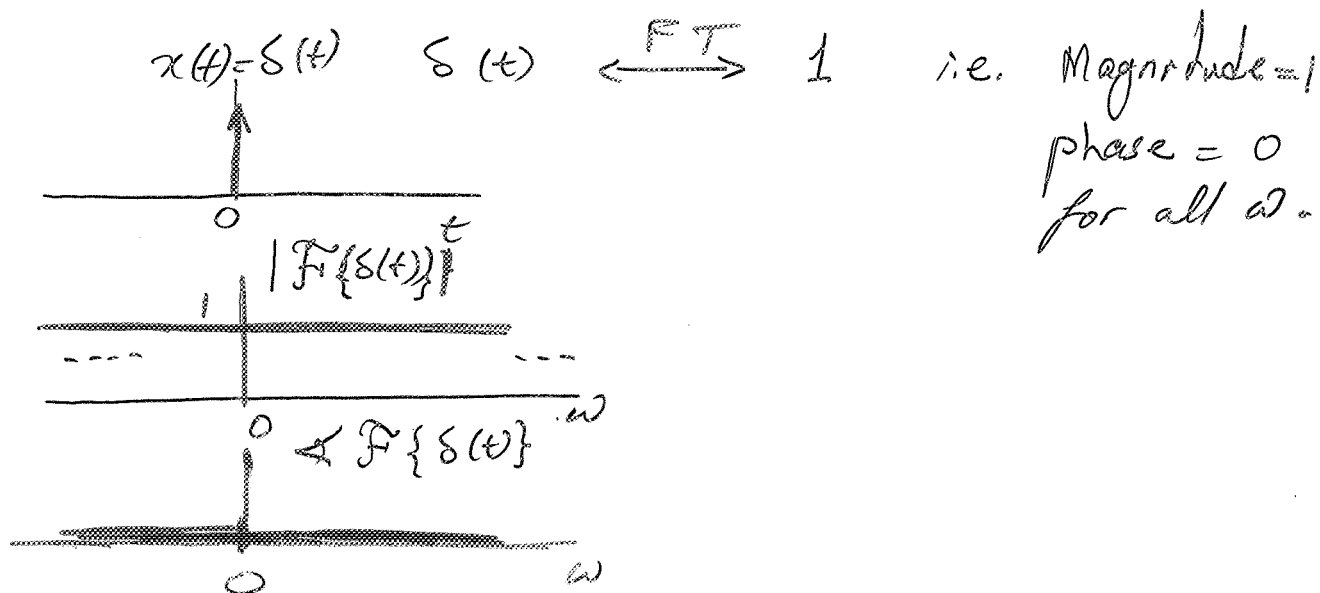
$$X(\omega) = \frac{-1}{j\omega + \alpha} \left\{ e^{-(j\omega + \alpha)t} \right\}_{t=0}^{\infty}$$

$$X(\omega) = \frac{-1}{j\omega + \alpha} \{ 0 - 1 \} = \frac{1}{j\omega + \alpha}$$

$$|X(\omega)| = \frac{1}{\sqrt{\alpha^2 + \omega^2}}, \quad \angle X(\omega) = -\left(\tan^{-1}\left(\frac{\omega}{\alpha}\right)\right)$$

Ex 4 For  $x(t) = \delta(t)$ , find  $X(\omega)$ .

$$X(\omega) = \mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$



Note

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$\delta(t) = \frac{1}{2\pi} \lim_{\alpha \rightarrow \infty} \int_{-\alpha}^{\alpha} e^{j\omega t} d\omega = \frac{1}{2\pi} \lim_{\alpha \rightarrow \infty} \left( \frac{1}{jt} \right) \left\{ e^{j\omega t} \right\}_{-\alpha}^{\alpha}$$

$$\delta(t) = \frac{1}{2\pi} \lim_{\alpha \rightarrow \infty} \left( \frac{1}{jt} \right) \left\{ e^{j\alpha t} - e^{-j\alpha t} \right\}$$

$$= \frac{1}{2\pi} \lim_{\alpha \rightarrow \infty} \left( \frac{2}{t} \right) \sin(\alpha t) =$$

$$\delta(t) = \frac{1}{\pi} \lim_{\alpha \rightarrow \infty} \left( \frac{\sin(\alpha t)}{t} \right).$$



Ex 5 For  $x(t) = A$ , find  $\mathcal{F}\{A\} = X(\omega)$ .

$$\begin{cases} X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \end{cases}$$

Solution

$$\hat{X}(\omega) = A \lim_{\alpha \rightarrow \infty} \int_{t=-\alpha}^{\alpha} e^{-j\omega t} dt$$

$$= A \lim_{\alpha \rightarrow \infty} \left( \frac{-1}{j\omega} \right) \left\{ e^{-j\omega t} \right\}_{t=-\alpha}^{t=\alpha}$$

$$= A \lim_{\alpha \rightarrow \infty} \left\{ \frac{2}{\omega} \sin(\omega \alpha) \right\}$$

$$\hat{X}(\omega) = A \delta(\omega)$$

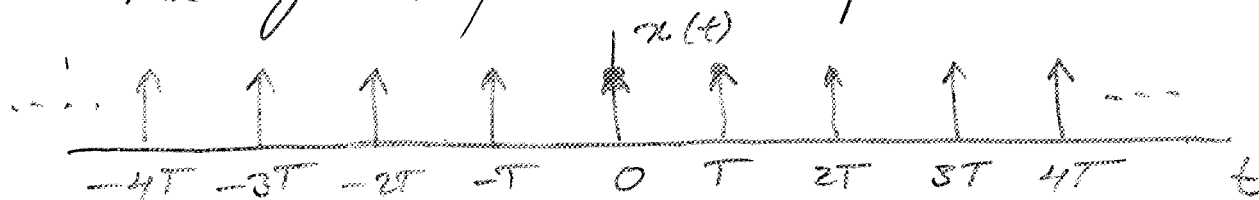
But  $\hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A \delta(\omega) e^{j\omega t} d\omega = \frac{A}{2\pi}$

To keep the FT pair one-to-one and correct with then let

$$X(\omega) = 2\pi A \delta(\omega)$$

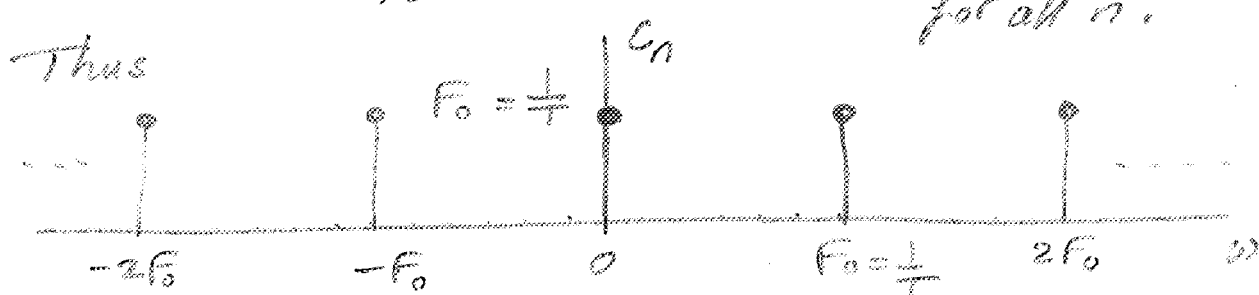
thus  $x(t) = A \xleftrightarrow{\text{FT}} A 2\pi \delta(\omega) = X(\omega)$

Ex 6 Find spectrum of  $x(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$ .  
 This signal is periodic with period  $T$ .



$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j \frac{2\pi n}{T} t} dt$$

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-j \frac{2\pi n}{T} t} dt = \frac{1}{T} = F_0 \quad \text{for all } n.$$



Now

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j \frac{2\pi n}{T} t} = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{j \frac{2\pi n}{T} t}$$

$$x(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{j \frac{2\pi n}{T} t}$$

Taking FT from both sides yields

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$X(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j(\omega - \frac{2\pi n}{T})t} dt$$

where

$$\int_{-\infty}^{\infty} e^{-j(\omega - \frac{2\pi n}{T})t} dt = \lim_{\alpha \rightarrow \infty} \int_{-\alpha}^{\alpha} e^{-j(\omega - \frac{2\pi n}{T})t} dt$$

$$\lim_{\alpha \rightarrow \infty} \int_{t=-\alpha}^{\alpha} e^{-j(\omega - \frac{2\pi n}{T})t} dt = \lim_{\alpha \rightarrow \infty} \left( \frac{2}{\omega - \frac{2\pi n}{T}} \right) \sin \left[ \left( \omega - \frac{2\pi n}{T} \right) \alpha \right]$$

$$= 2\pi \delta \left( \omega - \frac{2\pi n}{T} \right)$$

Thus

$$X(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta \left( \omega - \frac{2\pi n}{T} \right)$$

Note:

As before, we used

$$\delta(\omega) = \frac{1}{2\pi} \int_{t=-\infty}^{\infty} e^{j\omega t} dt$$

and

$$1 \xleftrightarrow{FT} 2\pi \delta(\omega)$$

Note

See Table 4.1 in pages 172, 173,  
for Fourier Transform pairs. Soliman Book.

Also, see

Pages 328 and 329 of Oppenheim book

## Properties of the Fourier Transform

(1) Linearity: (ie Superposition properties hold):

$$\text{If } x_1(t) \xrightarrow{\text{FT}} X_1(\omega)$$

$$\text{and } x_2(t) \longleftrightarrow X_2(\omega)$$

then

$$v(t) = a x_1(t) + b x_2(t) \longleftrightarrow V(\omega) = a X_1(\omega) + b X_2(\omega)$$

(2) Symmetry:

If  $x(t)$  is real-valued function

$$\text{then } X(-\omega) = X^*(\omega) \text{ or } X(\omega) = X^*(-\omega)$$

$$\underbrace{|X(\omega)| = |X(-\omega)|}_{\text{Even}} , \quad \underbrace{\phi(\omega) = -\phi(-\omega)}_{\text{odd}}$$

Magnitude spectrum

phase spectrum

If  $x(t)$  is real and even (ie  $x(t) = x(-t)$ )

then  $X(\omega)$  is real.

### (3) Time-Shifting:

If  $x(t) \longleftrightarrow X(\omega)$

then

$$v(t) = x(t - \alpha) \longleftrightarrow V(\omega) = e^{\mp j\omega\alpha} X(\omega)$$

### (4) Frequency-Shifting:

If  $x(t) \longleftrightarrow X(\omega)$

then

$$v(t) = e^{\pm j\omega_0 t} x(t) \longleftrightarrow V(\omega) = X(\omega \mp \omega_0)$$

### (5) Time-scaling:

If  $x(t) \longleftrightarrow X(\omega)$

then

$$v(t) = x(\alpha t) \longleftrightarrow V(\omega) = \frac{1}{|\alpha|} X\left(\frac{\omega}{\alpha}\right)$$

( $\alpha$  is a real constant)

because

$$V(\omega) = \int_{t=-\infty}^{\infty} v(t) e^{-j\omega t} dt = \int_{t=-\infty}^{\infty} x(\alpha t) e^{-j\omega t} dt$$

$$\text{let } \alpha t = \beta \quad dt = \frac{1}{\alpha} d\beta, \quad t = \frac{\beta}{\alpha}$$

$$V(\omega) = \frac{1}{|\alpha|} \int_{\beta=-\infty}^{\infty} x(\beta) e^{-j\frac{\omega}{\alpha}\beta} d\beta = \frac{1}{|\alpha|} X\left(\frac{\omega}{\alpha}\right)$$

For  $\alpha < 0$  or  $\alpha > 0$ .

If  $\alpha < 1$  then  $v(t) = x(\alpha t)$  is expanded version of  $x(t)$  in time domain.  
 and  $V(\omega) = \frac{1}{|\alpha|} X\left(\frac{\omega}{\alpha}\right)$  is scaled and contracted version of  $X(\omega)$

If  $\alpha > 1$  then

$v(t) = x(\alpha t)$  is contracted version of  $x(t)$  in time-domain.

and

$V(\omega) = \frac{1}{|\alpha|} X\left(\frac{\omega}{\alpha}\right)$  is scaled and expanded version of  $X(\omega)$ .

### (6) Differentiation in time:

If  $x(t) \leftrightarrow X(\omega)$

then

$$v(t) = \frac{d x(t)}{dt} \leftrightarrow V(\omega) = j\omega X(\omega)$$

and

$$g(t) = \frac{d^n x(t)}{dt^n} \leftrightarrow G(\omega) = (j\omega)^n X(\omega)$$

(7) Duality property of FT:

If  $x(t) \longleftrightarrow X(\omega) = \mathcal{F}\{x(t)\}$   
then

$$X(t) \longleftrightarrow 2\pi x(-\omega)$$

and if  $x(t) = x(-t)$  is an even function, then

$$X(t) \longleftrightarrow 2\pi x(\omega) \quad \text{Since } x(\omega) = x(-\omega).$$

Proof:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

or

$$2\pi x(t) = \int_{\omega=-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \Big|_{\omega=\alpha} = \int_{\alpha=-\infty}^{\infty} X(\alpha) e^{j\alpha t} d\alpha$$

For  $\omega = \alpha$

Now change  $t$  to  $-t$  on both sides, then

$$2\pi x(-t) = \int_{\alpha=-\infty}^{\infty} X(\alpha) e^{-j\alpha t} d\alpha$$

We now replace  $t$  by  $\omega$  to get

$$2\pi x(-\omega) = \int_{\alpha=-\infty}^{\infty} X(\alpha) e^{-j\alpha\omega} d\alpha$$

$\alpha$  is integration variable. Replace  $\alpha$  by  $t$  to get

$$2\pi x(-\omega) = \int_{t=-\infty}^{\infty} X(t) e^{-j\omega t} dt$$

That is

$$X(t) \xleftrightarrow{FT} 2\pi x(-\omega) = \mathcal{F}\{X(t)\}$$

□

Ex a

$$x(t) = \delta(t) \xleftrightarrow{FT} X(\omega) = \mathcal{F}\{\delta(t)\} = 1$$

$$\delta(t) \longleftrightarrow 1$$

then

$$1 \longleftrightarrow 2\pi \delta(\omega)$$

Ex b

$$y(t) = \delta(t - t_0) \longleftrightarrow e^{-j\omega t_0} = \gamma(\omega)$$

then

$$e^{j\omega t_0} \longleftrightarrow 2\pi \delta(\omega - \omega_0)$$

Ex c

$$x(t) = \cos(\omega_0 t) = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$$

Now we know (from Ex b above) that

$$e^{j\omega_0 t} \longleftrightarrow 2\pi \delta(\omega - \omega_0)$$

and

$$e^{-j\omega_0 t} \longleftrightarrow 2\pi \delta(\omega + \omega_0)$$

Thus

$$x(t) = \cos(\omega_0 t) \longleftrightarrow \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)$$

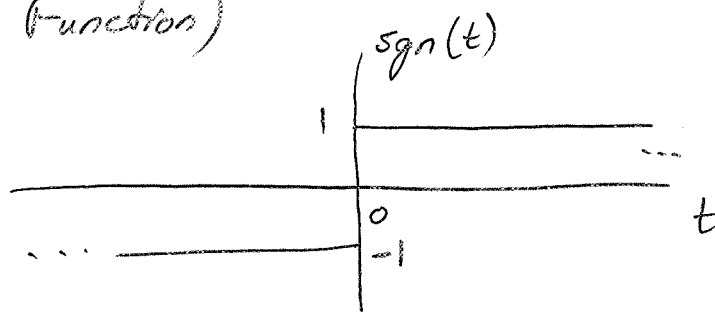
Note

$$y(t) = \sin(\omega_0 t) \longleftrightarrow j\pi \delta(\omega + \omega_0) - j\pi \delta(\omega - \omega_0)$$



Ex. d (The Signum Function)

$$\text{sgn}(t) = \begin{cases} -1, & t < 0 \\ 0, & t = 0 \\ 1, & t > 0 \end{cases}$$

Solution

We can use the time differentiation property of FT to find spectrum of the  $\text{sgn}(t)$ , as follows.

Recall that if  $x(t) \longleftrightarrow X(\omega)$   
 then  $\frac{d x(t)}{d t} = x'(t) \longleftrightarrow j\omega X(\omega)$

Now

$$\frac{d \text{sgn}(t)}{d t} = 2 \delta(t), \text{ Thus}$$

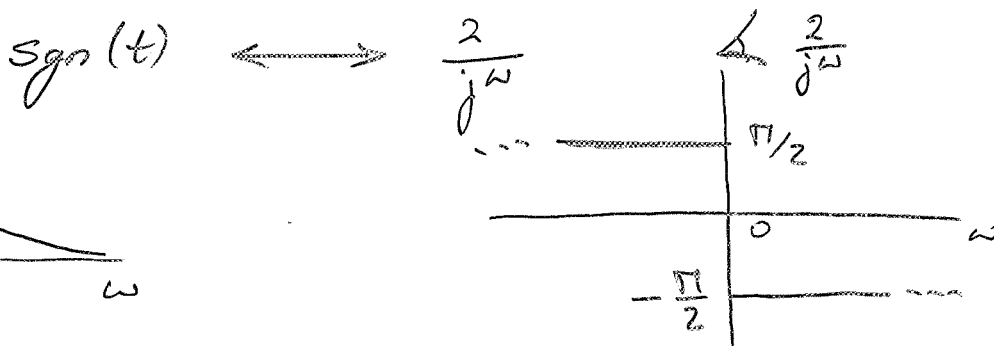
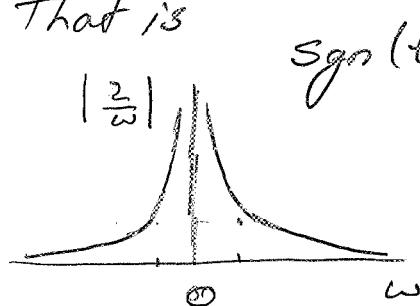
$$\frac{d \text{sgn}(t)}{d t} = 2 \delta(t) \xrightarrow{FT} j\omega \mathcal{F}\{2 \delta(t)\} = 2$$

Thus

$$2 = j\omega \mathcal{F}\{\text{sgn}(t)\} \text{ and}$$

$$\mathcal{F}\{\text{sgn}(t)\} = \frac{2}{j\omega}$$

That is



Note Let  $y(t) = \int_{-\infty}^t x(\tau) d\tau$

and

$$y(t) \longleftrightarrow Y(\omega)$$

$$x(t) \longleftrightarrow X(\omega)$$

then

$$v(t) = \frac{dy(t)}{dt} = x(t) \longleftrightarrow j\omega Y(\omega) = X(\omega)$$

and

$$Y(\omega) = \frac{1}{j\omega} X(\omega)$$

The above is true if  $\int_{-\infty}^{\infty} x(t) dt = X(0) = 0$   
 or  $y(\infty) = 0$ . That is when  $x(t)$  has zero-mean or no DC value (i.e.  $X(0) = 0$ ).

If  $X(0) \neq 0$ , i.e. when  $x(t)$  has a dc value then

$$y(t) = \int_{-\infty}^t x(\tau) d\tau \longleftrightarrow \pi X(0) \delta(\omega) + \frac{1}{j\omega} X(\omega)$$

Ex.e Let  $x(t) = u(t)$  (step function).

Now

$$x(t) = u(t) = \frac{1}{2} + \frac{1}{2} \text{sgn}(t)$$

Thus

$$\begin{aligned} \mathcal{F}\{u(t)\} &= \mathcal{F}\left\{\frac{1}{2}\right\} + \frac{1}{2} \mathcal{F}\{\text{sgn}(t)\} = \\ &= \pi \delta(\omega) + \frac{1}{j\omega} \end{aligned}$$

Ex. f

$$x(t) = \cos(\omega_0 t) u(t) = h(t) \cdot u(t)$$

where

$$h(t) = \cos(\omega_0 t) \longleftrightarrow \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0) = H(\omega)$$

and that

$$u(t) \longleftrightarrow \pi \delta(\omega) + \frac{1}{j\omega} = U(\omega)$$

If we use FT convolution property, then

$$x(t) = h(t) \cdot u(t) \longleftrightarrow X(\omega) = H(\omega) * U(\omega)$$

that is

$$X(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(\lambda) H(\omega - \lambda) d\lambda$$

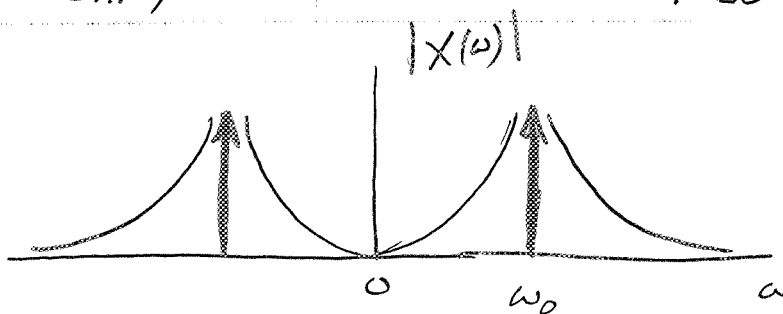
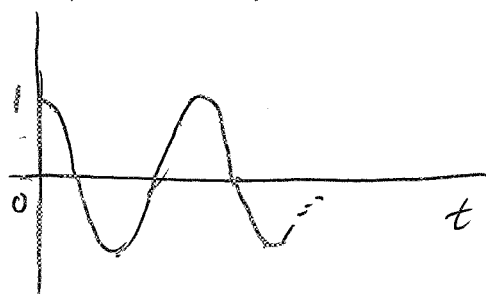
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \pi \delta(\lambda) + \frac{1}{j\lambda} \right\} \left\{ \pi \delta(\omega - \lambda - \omega_0) + \pi \delta(\omega - \lambda + \omega_0) \right\} d\lambda$$

$$X(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \pi^2 \delta(\lambda) \delta(\omega - \lambda - \omega_0) + \pi^2 \delta(\lambda) \delta(\omega - \lambda + \omega_0) + \frac{\pi}{j\lambda} \delta(\omega - \lambda - \omega_0) + \frac{\pi}{j\lambda} \delta(\omega - \lambda + \omega_0) \right\} d\lambda$$

$$X(\omega) = \frac{1}{2\pi} \left\{ \pi^2 \delta(\omega - \omega_0) + \pi^2 \delta(\omega + \omega_0) + \frac{\pi}{j(\omega - \omega_0)} + \frac{\pi}{j(\omega + \omega_0)} \right\}$$

$$X(\omega) = \frac{\pi}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{-j\omega}{\omega^2 - \omega_0^2}$$

$$x(t) = \cos(\omega_0 t) u(t)$$



□

\* See/Read Example 4.3.13, page 186 of the Textbook.

\* See Table 4.2, page 189 of the Textbook for properties of Fourier Transform.

### Energy of Aperiodic Signals

Energy of  $x(t)$  (if  $x(t)$  is energy signal) is given by

$$E_x = \int_{t=-\infty}^{\infty} |x(t)|^2 dt = \int_{t=-\infty}^{\infty} x(t) x^*(t) dt$$

but  $x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$

and  $x^*(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X^*(\omega) e^{-j\omega t} d\omega$

Thus

$$E_x = \int_{t=-\infty}^{\infty} x(t) \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X^*(\omega) e^{-j\omega t} d\omega dt$$

$$= \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X^*(\omega) \underbrace{\left[ \int_{t=-\infty}^{\infty} x(t) e^{-j\omega t} dt \right]}_{X(\omega)} d\omega$$

$$E_x = \int_{t=-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X^*(\omega) X(\omega) d\omega$$

$$E_x = \int_{t=-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$$|X(\omega)|^2 \triangleq S_{xx}(\omega) = S_{xx}(-\omega) \text{ is called}$$

Energy Density function (or spectrum) of  $x(t)$ .

$S_{xx}(\omega)$  represents distribution of signal energy with  $\omega$ .

Energy of  $x(t)$  over any frequency interval  $\omega_1 \leq \omega \leq \omega_2$  is obtained by

$$E = \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} |X(\omega)|^2 d\omega$$

Note The relation

$$E_x = \int_{t=-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} |X(\omega)|^2 d\omega$$

or for  $\omega = 2\pi F$ ,  $d\omega = 2\pi dF$ :

$$= \int_{F=-\infty}^{\infty} |X(F)|^2 dF$$

is the Parseval's theorem or Relation showing preservation of energy between the signal energy in time and frequency domains.

Recall that for a periodic signal  $y(t) = y(t+T)$  where  $T$  and  $F_0 = \frac{1}{T}$  are fundamental period and harmonics, respectively, we had

$$y(t) = y(t+T) = \sum_{k=-\infty}^{\infty} C_k e^{j \frac{2\pi k}{T} t}$$

$$C_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} y(t) e^{-j \frac{2\pi k}{T} t} dt$$

and that

$$P_{av.} = \frac{1}{T} \int_0^T |y(t)|^2 dt = \sum_{k=-\infty}^{\infty} |C_k|^2$$

is Parseval's Relation.

And that  $|C_k|^2$  is called Power Density function or Power Density spectrum or simply power spectrum of  $y(t) = y(t+T)$ .

Now for a power signal (not periodic),  $v(t)$ , we have:

$$\begin{aligned} P_{\text{Average power}} &= \lim_{T \rightarrow \infty} \left[ \frac{1}{2T} \int_{-T}^T |v(t)|^2 dt \right] \\ &= \lim_{T \rightarrow \infty} \left[ \frac{1}{2T} \int_{-\infty}^{\infty} |v_T(t)|^2 dt \right] \quad (I) \end{aligned}$$

where  $v_T(t) = \begin{cases} v(t) & \text{for } -T < t < T \\ 0 & \text{elsewhere} \end{cases}$

Using Parseval's relation, we can write Equation (I) above as

$$\begin{aligned} P &= \frac{1}{2\pi} \lim_{T \rightarrow \infty} \left[ \frac{1}{2T} \int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega \right] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \left[ \frac{|X_T(\omega)|^2}{2T} \right] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega \end{aligned}$$

where  $S(\omega) = \lim_{T \rightarrow \infty} \left[ \frac{|X_T(\omega)|^2}{2T} \right]$

$S(\omega)$  is called the Power Density spectrum or simply Power spectrum of the signal  $v(t)$ .